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Hamiltonian cycles in bipartite toroidal graphs with a partite set of degree four vertices

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ABSTRACT

Let G be a 3-connected bipartite graph with partite sets $X \cup Y$ which is embeddable in the torus. We shall prove that G has a Hamiltonian cycle if (i) G is *balanced*, i.e., $|X| = |Y|$, and (ii) each vertex $x \in X$ has degree four. In order to prove the result, we establish a result on orientations of quadrangular torus maps possibly with multiple edges. This result implies that every 4-connected toroidal graph with toughness exactly one is Hamiltonian, and partially solves a well-known Nash-Williams' conjecture.

1. Introduction

A *surface* is a connected compact 2-dimensional manifold without boundary. A *map* on a surface F^2 means a fixed embedding of a graph on F^2. A *Hamiltonian* cycle of a graph G is a cycle passing through all vertices of G exactly once. A graph G is said to be *Hamiltonian* if G has a Hamiltonian cycle. We would like to consider whether graphs on surfaces have Hamiltonian cycles.

Whitney proved that any 4-connected plane triangulation is Hamiltonian [19], and Tutte improved this result for 4-connected plane graphs [18]. Starting from these results, Hamiltonicity of graphs on surfaces has been extensively studied, for example, Thomas and Yu proved that every 4-connected projective plane map is Hamiltonian [14]. Note that in those theorems, the 4-connectedness cannot be omitted, since some non-4-connected graphs on those surfaces have no Hamiltonian cycles. So next, we are interested in the toroidal case. Actually, Grünbaum and Nash-Williams posed the following conjecture.

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Conjecture 1. *(See Grünbaum [7] and Nash-Williams [10].) Every 4-connected torus map is Hamiltonian.*

This conjecture has attracted much attention for long years, but it is still open. Actually, many researchers gave some partial solutions to Conjecture 1. Barnette [4], and Brunet and Richter [5] showed that every 5-connected toroidal triangulation has a Hamiltonian path, and a Hamiltonian cycle, respectively. Thomas and Yu improved these results and showed that every 5-connected toroidal map is Hamiltonian [15], and later, Thomas, Yu and Zang showed the existence of Hamiltonian paths in 4-connected toroidal graphs [16]. Note that the proofs of the above results used the concept of a "Tutte path or cycle". For more information about it, see Section 3 of [6]. As mentioned later, we also give a partial solution to Conjecture 1 in this paper, but our method is very different from those using a "Tutte path or cycle".

A *quadrangulation* on a surface F^2 is a map of a simple graph on F^2 such that each face is bounded by a 4-cycle. Altshuler proved that every 4-connected torus quadrangulation has a Hamiltonian cycle [2]. It is easy to see that a torus quadrangulation is 4-connected if and only if it is 4-regular, and there is a simple standard form of 4-regular torus quadrangulations with a rectangular grid, which helps us to find Hamiltonian cycles in them. (Similarly, the Hamiltonicity of 6-regular torus triangulations is verified in the same paper by using a standard form of them [3]. Alspach and Zhang showed that the dual of such graphs is also Hamiltonian using algebraic approach [1].)

In this paper, restricting bipartite quadrangulations but relaxing the 4-connectivity, we shall prove the following.

Theorem 2. *Let G be a 3-connected bipartite torus quadrangulation. If one of the two partite sets consists only of degree four vertices, then G is Hamiltonian.*

Let us consider the condition of G to be a quadrangulation in the above theorem. Let G be a 3-connected bipartite graph on the torus with partite sets X and Y, and assume that every vertex in X has degree 4. Then by Euler's formula, G is a quadrangulation if and only if $|E(G)| = 2|V(G)|$. On the other hand, since $|E(G)| = \sum_{x\in X} \deg(x) = 4|X|$, it follows that G is *balanced* (i.e., $|X| = |Y|$) if and only if G is a torus *quadrangulation* (i.e., $|E(G)| = 2|V(G)|$). Hence, exchanging these two conditions in Theorem 2, we obtain Theorem 3, which is equivalent to Theorem 2 but will be more appealing, since the balance of X and Y is a trivial necessary condition for G to be Hamiltonian.

Theorem 3. *Let G be a 3-connected balanced bipartite graph which is embeddable in the torus. If one of the two partite sets consists only of degree four vertices, then G is Hamiltonian.*

Theorem 3 generalizes Altshuler's theorem [2] in the bipartite case since, as mentioned above, 4-connected torus quadrangulation is 4-regular. In Section 2, we will make more remarks on Theorems 2 and 3, and in Sections 3–5 we will prove Theorem 2.

A cycle C of length k is called a *k-cycle*. If k is even (resp., odd), then C is said to be *even* (resp., *odd*). A simple closed curve l on a non-spherical surface F^2 is said to be *essential* if l does not bound a 2-cell on F^2. An *essential cycle* of a graph on F^2 is a cycle whose edges induce an essential curve. A *k-vertex* (resp., a *k-face*) is a vertex (resp., face) of degree exactly k.

The *representativity* of a map G on a non-spherical surface F^2, denoted $r(G)$, is the minimum number of intersecting points of G and γ, where γ ranges over all essential simple closed curves on F^2. We say that G is *k-representative* if G has representativity at least k.

Let H be a map on a surface F^2 and we suppose that all vertices of H are colored by black. The *face subdivision* of H, denoted $\tilde{H}$, is obtained from H by adding a new white vertex to each face of H and joining it to all vertices lying on the corresponding face boundary. The *radial graph* of H, denoted $R(H)$, is obtained from $\tilde{H}$ by removing all edges joining two black vertices. It is easy to see that $R(H)$ is bipartite and each face of $R(H)$ is quadrilateral. Moreover, $R(H)$ is simple if and only if each face of H is bounded by a cycle.

We say that a graph G is *k-tough* if $\frac{|S|}{\omega(G-S)} \geqslant k$ for any $S \subset V(G)$ with $G - S$ disconnected, where $\omega(\cdot)$ is the number of components. The *toughness* of G is defined to be the maximum real number k such that G is k-tough. It is well known that if G is Hamiltonian, then G is 1-tough.

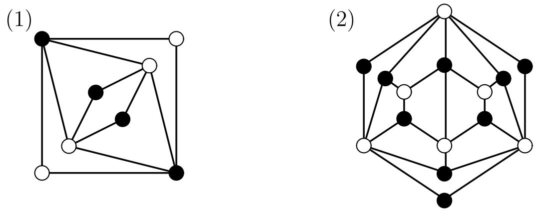

Fig. 1. Plane quadrangulations S and T.

2. Remarks on Theorems 2 and 3

In this section, we make four remarks on Theorems 2 and 3.

3-Connectedness and a partite set of 4-vertices. Let us consider whether the conditions in Theorem 2 can be omitted. Let K be a 4-regular bipartite torus quadrangulation, which must be balanced with black and white vertices. Let S and T be the two bipartite plane graphs shown in Fig. 1(1) and (2), respectively. Let T' be the graph T with white and black interchanged.

We first consider the 3-connectedness. Let f be a face of K. Let G be the bipartite torus quadrangulation obtained from K and S by pasting the boundary 4-cycles of f and that of S so that vertices with the same color are identified. Then G is a bipartite torus quadrangulation in which all white vertices are of degree four, but it is not 1-tough, since G with the two white inner vertices of S removed has three components. Hence, if we omit the 3-connectedness of the graph, then Theorem 2 does not hold. (Note that using other constructions, we can show that Theorem 2 does not hold even if we assume 2-connectedness and "minimum degree at least three", instead of the 3-connectedness.)

We secondly consider the condition for a partite set of 4-vertices. Let e and e' be edges of K which do not lie on the boundary 4-cycle of the same face, and let $K' = K - \{e, e'\}$, where we let h and h' be the hexagonal faces of K' containing e and e' in K, respectively. Let Q be the bipartite torus quadrangulation obtained from K' by pasting T to h and T' to h' so that vertices with the same color coincide. Since we added the same number of black and white vertices, G is a balanced 3-connected bipartite torus quadrangulation, but it is not 1-tough. (Removing the six white vertices of T in G, we get at least seven components.) Since G is not Hamiltonian, the condition for a partite set of 4-vertices cannot be omitted, either.

1-Toughness and embeddability in the torus. Let G be any 4-connected graph on any surface F^2 with Euler characteristic $\chi(F^2) \geqslant 0$. Then G must be 1-tough. (See [12] for the argument.) On the other hand, if we relax the 4-connectivity, those surfaces admit infinitely many 3-connected balanced bipartite graphs which are not 1-tough, as in the previous section. However, restricting the vertex degree in one of the partite sets to be four in 3-connected balanced bipartite toroidal graphs, we get the Hamiltonicity of them as in Theorem 3. Therefore, the condition of G in Theorem 3 is sufficient for G to be 1-tough, though it seems difficult to prove it directly from the assumption on G. So Corollary 4 is worth mentioning.

Corollary 4. *Let G be a 3-connected balanced bipartite graph which is embeddable in the torus. If one of the two partite sets consists only of degree four vertices, then G is 1-tough.*

Here we show that the condition on G to be embeddable in the torus is necessary in the above corollary (and also in Theorem 3). Let A be three vertices in the same partite set of $K_{3,3}$. Let B be three vertices in the same partite set of a 4-regular bipartite quadrangulation T on the torus which has a unique embedding on the torus. (Such a quadrangulation T exists, see [11].) Let G be the graph obtained by joining A and B with a perfect matching. Then G is a 3-connected balanced bipartite graph one of whose partite sets consists only of degree four vertices, but G is not embeddable in the torus since T has a unique embedding on the torus. It is easy to see that G is not 1-tough since $G - A$ has four components.

4-Connected non-Hamiltonian maps of negative Euler characteristics. Let Q be a quadrangulation on a surface with a negative Euler characteristic. Then, by Euler's formula, the number of faces of Q is strictly greater than that of vertices, and hence the face subdivision $\tilde{Q}$ of Q is not 1-tough. Therefore, $\tilde{Q}$ is 4-connected but it is not Hamiltonian. Hence, for maps on surfaces with negative Euler characteristics, the 4-connectedness does not necessarily imply the Hamiltonicity. (Yu proved that every 5-connected triangulation on any non-spherical surface with sufficiently large representativity is Hamiltonian [20].)

If we consider the above construction for the torus, then we can get a 4-connected graph on the torus with the toughness exactly 1. (The existence of 4-connected graphs on the torus having the toughness exactly one is a big reason for the difficulty of Conjecture 1.) So Theorem 2 asserts that the face subdivision of any torus quadrangulation has a Hamiltonian cycle, though its toughness is exactly 1.

Hamiltonicity of 4-connected toroidal graphs with toughness exactly one. As mentioned before, there exist 4-connected toroidal graphs with toughness exactly one, which seems to make the problem difficult. However, using Theorem 3, we can prove the following surprising result:

Corollary 5. *Let G be a 4-connected toroidal graph. If the toughness of G is exactly one, then G is Hamiltonian.*

Proof. Let G be a 4-connected toroidal graph with the toughness exactly one. Then there exists $S \subset V(G)$ such that $\omega(G-S) = |S|$. Here we take S so that $|S|$ is as small as possible. Let H be the torus map obtained from G by contracting a spanning tree of each component of $G - S$ into a single vertex, deleting all but one of the edges in multiple edges bounding a 2-cell, deleting all the created loops, and deleting each edge joining two vertices in S. Note that H is a balanced bipartite graph of order $|S| + \omega(G - S)$. Since G is 4-connected, we have $|E(H)| \geqslant 4\omega(G - S)$. Moreover, since G is embeddable in the torus, we have $|E(H)| \leqslant 2(|S| + \omega(G - S))$, by Euler's formula. By the fact "$\omega(G - S) = |S|$", the equalities must hold in the above two inequalities. So any vertex in H corresponding to a component of $G - S$ has degree four, and H is a quadrangular map on the torus.

Let C be a component of $G - S$ and let c be the vertex of H corresponding to C. Assume that C is not contained in a 2-cell. Then C contains an essential cycle, which represents an essential curve of the torus intersecting H only at c. Since H is a quadrangular map, there exists a quadrangular face cu_1cu_2c which contains that essential curve. (Note that both of the pairs (c, u_1) and (c, u_2) are joined by multiple edges which do not bound a 2-cell.) Since c has degree exactly four, u_1 and u_2 are the only neighbors of C in G, contradicting the 4-connectedness of G. Therefore, C is contained in a 2-cell. In particular, no vertex in S is joined to c by multiple edges (which do not bound a 2-cell) in H, because there are at least four neighbors of C in G. Moreover, we can take a 2-cell containing C so that it does not contain S, since otherwise at most three vertices of S around C form a cutset of G.

Suppose that H has two distinct vertices u_1, u_2 with $H - \{u_1, u_2\}$ disconnected. Since H is a quadrangular map, H has two faces, say u_1au_2x and u_1bu_2y such that a and b are inner vertices of the plane subgraph bounded by the 4-cycle u_1xu_2y. Observe that either $u_1, u_2 \in S$ or $u_1, u_2 \notin S$. In the former case, $G - \{u_1, u_2\}$ is disconnected, contrary to the 4-connectedness of G. In the latter case, we have $a, b, x, y \in S$. If $a \neq b$, then H has two faces xu_1yc and xu_2yd for some $c, d \in V(H)$, since $\deg_H(u_1) = \deg_H(u_2) = 4$. Then $G - \{x, y\}$ is disconnected, contrary to the 4-connectedness of G. On the other hand, if $a = b$, then putting $S' = S - \{a\}$, we have $\omega(G - S') = \omega(G - S) - 1 = |S| - 1 = |S'|$, contrary to the minimality of S. Hence H is 3-connected.

By Theorem 3, H has a Hamiltonian cycle T. We shall recover each component of $G - S$ and find a Hamiltonian cycle of G.

Let $v \notin S$ be a vertex in H and let u_1, u_2, u_3, u_4 be four distinct neighbors of v in H lying in this cyclic order on the torus. Let C_v be a component of $G - S$ corresponding to v, and let R_v be the graph obtained from the subgraph of G induced by $V(C_v) \cup \{u_1, u_2, u_3, u_4\}$ by adding $u_i u_{i+1}$ for $i = 1, 2, 3, 4$ if $u_i u_{i+1} \notin E(G)$ and deleting $u_1 u_3$ if $u_1 u_3 \in E(G)$. As mentioned before, R_v is a plane graph. Thomassen's result (Main Theorem in [17]) and Thomas and Yu's result (Lemma (2.4) in [14]) imply that for any distinct $i, j \in \{1, 2, 3, 4\}$, $R_v - \{u_k, u_l\}$ has a Hamiltonian path T_v connecting u_i and u_j, where $\{k, l\} = \{1, 2, 3, 4\} - \{i, j\}$. (Note that we use Thomassen's result for the case where $\{i, j\} = \{1, 3\}$ or $\{i, j\} = \{2, 4\}$, and Thomas and Yu's result for other cases.) Hence, when T passes through v_i, v, v_j in this order in H, then we can take a corresponding path T_v in G. Replacing all vertex $v \notin S$ in T with T_v, we obtain a Hamiltonian cycle of G. $\square$

3. 2-Orientations and vertex-faces curves

Let Q be a quadrangulation and let f be a face of Q bounded by a cycle $abcd$. We say that each of $\{a, c\}$ and $\{b, d\}$ is a *diagonal pair* of f in Q. A *quadrangular map* is a map, possibly with multiple edges, such that each face is bounded by a 4-cycle, and in particular, it is a quadrangulation if the graph is simple. It is known that any quadrangular map is 2-connected and 2-representative.

Proposition 6. *Let G be a bipartite torus quadrangulation with black and white vertices in which every white vertex has degree 4. Then there exists a quadrangular map Q such that $R(Q) = G$. In particular, if G is 3-connected, then Q has no contractible 2-cycle.*

Proof. First, let Q be the graph on the black vertices obtained from G in such a way that for each face of G, we connect two black vertices which form a diagonal pair of the face. Then $R(Q) = G$. Q has no loop and every face of Q is bounded by a cycle because G is by definition simple. In particular, every face of Q is bounded by a cycle and can be taken as a quadrilateral since each white vertex of Q has degree four.

Suppose that Q has a contractible 2-cycle $C = xy$. Then G has two quadrilateral faces, say $xpyq$ and $xsyt$, each of whose diagonal pair is $\{x, y\}$. Since C is contractible on the torus, we can take a contractible simple closed curve γ along C intersecting G only at x and y. Hence $G - \{x, y\}$ is disconnected and hence G is not 3-connected. $\square$

Let Q be a map on a surface F^2 and let $\mathcal{L}_m = \{l_1, \dots, l_m\}$ be a set of disjoint simple closed curves on F^2. We say that $\mathcal{L}_m$ is a *vertex-face m-family for* Q if for each vertex or face x, there exists an integer i with $1 \leqslant i \leqslant m$ such that l_i visits x exactly once, x is not passed by l_j for any $j \neq i$, and every l_i crosses no edge of Q transversely. In particular, when $m = 1$, the unique element of $\mathcal{L}_1$ is called a *vertex-face curve* for Q.

The following is an easy observation, since a vertex-face curve passes through vertices and faces alternately.

Proposition 7. *Let Q be a map on a surface and let $R(Q)$ be the radial graph of Q. Then $R(Q)$ is Hamiltonian if and only if Q admits a vertex-face curve.*

Hence, by Propositions 6 and 7, the following theorem implies Theorem 2. So, the main purpose of this paper is to show Theorem 8.

Theorem 8. *Every quadrangular map Q on the torus with no contractible 2-cycle admits a vertex-face curve.*

In [9], Theorem 8 has already been proved when Q is simple and bipartite. So, in this paper, modifying the argument, we prove Theorem 8 when Q is non-bipartite.

In order to prove Theorem 8, we use an *orientation* of a graph G, that is, an assignment of a direction to each edge of G. Let $\vec{G}$ denote the graph with the orientation and distinguish it from the undirected graph G. For a vertex v of $\vec{G}$, the *outdegree* of v is the number of directed edges outgoing from v and denoted by $od(v)$. We say that $\vec{G}$ is a *k-orientation* or *k-oriented* if each vertex of $\vec{G}$ has outdegree exactly k. We need the following.

Lemma 9. *Every quadrangular map with no contractible 2-cycle on the torus admits a 2-orientation.*

The above has been proved only for quadrangulations in [9], and we modify the argument to deal with quadrangular map possibly with multiple edges. Let G be a graph and let $f : V(G) \to \{0, 1, 2, \ldots\}$ be a function. An orientation of G is called an f-orientation if $od(v) = f(v)$ for any $v \in V(G)$. The following proposition gives a necessary and sufficient condition for a graph to have an f-orientation. For $S \subset V(G)$, let $[S]$ denote the subgraph of G induced by S.

Proposition 10. *(See [9].) Let G be a graph and let $f : V(G) \to \{0, 1, 2, \ldots\}$ be a function. Then G has an f-orientation $\vec{G}$ if and only if $\sum_{v\in V(G)} f(v) = |E(G)|$, and for any $S \subset V(G)$ with $[S]$ connected, $\sum_{s\in S} f(s) \geqslant |E([S])|$.*

Note that in [9], we dealt with all $S \subset V(G)$, but it is enough to consider only all $S \subset V(G)$ with $[S]$ connected, since if $[S]$ is disconnected, then we can consider each component of $[S]$ separately.

We shall prove Lemma 9, using Proposition 10.

Proof of Lemma 9. By Proposition 10, a graph Q with n vertices has a 2-orientation if and only if (1) $|E(Q)| = 2n$, and (2) for any connected subgraph T of Q, $|E([T])| \leqslant 2|V(T)|$. By Euler's formula, we can verify (1) for Q.

We can deal with T as a torus map. Observe that T has no loop, since Q is 2-connected and 2-representative. Moreover, T has no face bounded by a 2-cycle by the assumption. Note that every cycle in Q that bounds a 2-cell must be even because the subgraph of Q inside the 2-cell must be bipartite. Thus T has no triangular face. Hence T has no k-face with $k \leqslant 3$, and so we have $2|E(T)| \geqslant 4|F(T)|$. By Euler's formula, $|V(T)| - |E(T)| + |F(T)| \geqslant 0$. By these two, we have $2|V(T)| \geqslant |E(T)|$ and we are done. □

In our proof, we need the following lemma.

Lemma 11. *Every quadrangular map on the torus has an essential even cycle.*

Let Q and H be two maps on the same non-spherical surface F^2. We say that H is a *surface-minor* of Q if H is obtained from Q by deletions and contractions of edges on F^2. Let Q be a k-representative map on a non-spherical surface. We say that Q is *k-minimal* if $r(Q) = k$ and $r(Q') < k$ for any proper surface-minor Q' of Q. It is known that for any non-spherical surface F^2 and any fixed integer $k \geqslant 1$, there exist only finitely many k-minimal maps on F^2, up to homeomorphism, see [13]. In particular, the complete list for 2-minimal torus maps has been determined in [8].

Theorem 12. *(See Nakamoto [8].) There exist exactly seven 2-minimal map on the torus, which are $T_1, \ldots, T_7$ listed in Fig. 2, in which each rectangle expresses the torus by identifying the top and the bottom, and the right and the left sides, respectively.*

Proof of Lemma 11. Let Q be a quadrangular map on the torus. Since Q is 2-representative, Q can be transformed by deletions and contractions of edges into one of $T_1, \ldots, T_7$ in Fig. 2, by Theorem 12. Let B be a torus map with a single vertex v and three essential pairwise non-homotopic loops e_1, e_2, e_3. We can verify that each T_i has B as a surface-minor. Hence, each T_i has an essential cycle contracted to e_j, for $j = 1, 2, 3$, since contractions of edges preserve homotopy types of cycles on the surface. Moreover, since some T_i is a surface-minor of Q, Q also has an essential cycle C_j contracted to e_j, for $j = 1, 2, 3$.

It is easy to see that in a quadrangular map, two homotopic closed walks have the same parity of length. Suppose that C_1 and C_2 have odd length. (Otherwise we are done.) Now cutting the torus where B embeds along e_1 and e_2, we get a rectangle with a diagonal e_3. Hence e_3 is homotopic to the concatenation of e_1 and e_2. Since C_1 and C_2 are non-homotopic on the torus, then they have a common vertex v in Q. Let W be a closed walk starting at v, proceeding along C_1 and return to v,

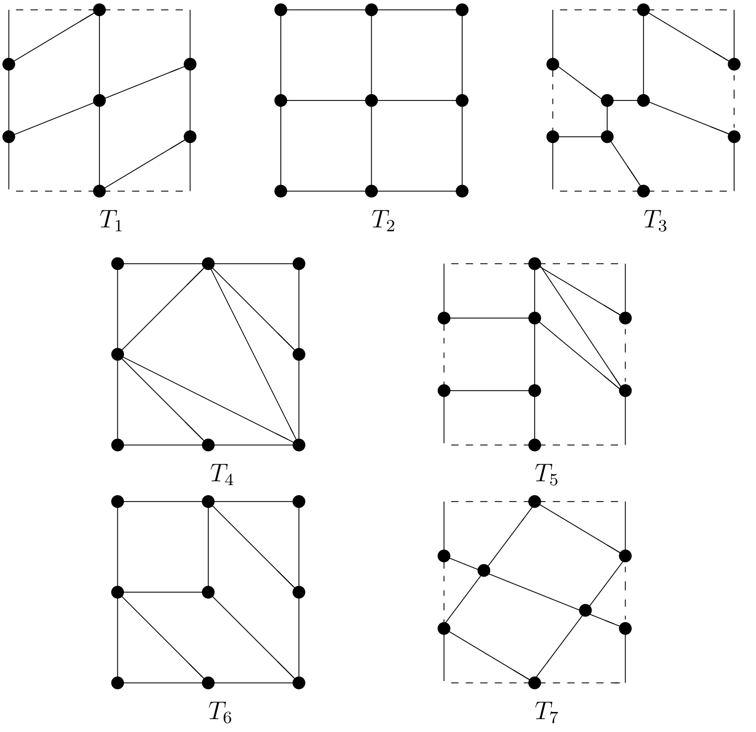

Fig. 2. The 2-minimal maps on the torus.

and proceeding along C_2 and stopping at v. Then W is an essential closed walk of even length, since C_1 and C_2 are non-homotopic essential odd cycles. Clearly, C_3 is an essential cycle homotopic to W, and hence its length must be even. $\square$

4. Converting Q into a bipartite graph

In this paper, we shall prove Theorem 8. In order to do so, we often cut the torus along a simple essential closed curve l to convert a non-bipartite torus quadrangulation into a bipartite torus quadrangulation. Let G be a map on a surface F^2 and suppose that each edge of G lies on l or intersects l at its endpoints. *Cutting G along l* is to cut F^2 along l so that in the map on the resulting surface, each vertex and each edge of G lying on l appears twice on the boundary. When F^2 is the torus, the map after cutting is one on the annulus.

Let Q be a non-bipartite torus quadrangular map. Then, by Lemma 11, Q has an essential even cycle $C = v_1 v_2 \ldots v_p$. Let Q_C be the annulus quadrangular map obtained from Q by cutting along C. Let $Y = y_1 y_2 \ldots y_p$ and $Z = z_1 z_2 \ldots z_p$ be the two boundary cycles of Q_C, where y_i and z_i correspond to v_i in Q for each i. Since Q is a quadrangular map and C is an essential even cycle, Q_C is a bipartite graph. Since Q is not bipartite and C is an even cycle, y_i and z_i are opposite colors. Let Q'_C be the bipartite torus quadrangular map naturally obtained from Q_C by adding the edges $y_i z_i$ for $i = 1, \ldots, p$.

In order to prove Theorem 8, the following is the most important argument in this paper, which will be proved in the last section.

Theorem 13. *Let Q be a non-bipartite quadrangular map on the torus with no contractible 2-cycle. Then an essential even cycle $C = v_1 \ldots v_p$ can be chosen in Q so that the bipartite quadrangular map Q'_C has a 2-orientation which has oriented edges*

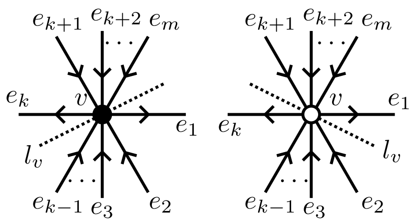

Fig. 3. Segments for vertices.

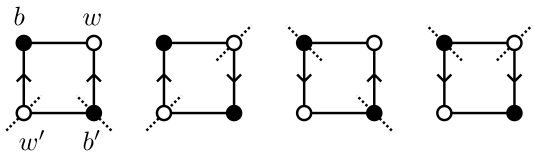

Fig. 4. Segments for faces.

- $y_{i+1}y_i$, z_iz_{i+1} *for any integer* i (*the index is taken modulo* p),
- z_iy_i *for any odd integer* i, *and*
- y_iz_i *for any even integer* i.

Before proceeding to the next section, we show that Theorem 13 implies Theorem 8, as follows.

Proposition 14. *If Theorem 13 holds, then so does Theorem 8.*

Proof. Let Q be a quadrangular map on the torus with no contractible 2-cycle. If Q is bipartite, then let $G = Q$. Otherwise we take an essential even cycle C and let $G = Q'_C$. By Lemma 9 (when $G = Q$) or by Theorem 13 (when $G = Q'_C$), G has a 2-orientation and let $\vec{G}$ be the oriented graph. When Q is non-bipartite, we take a 2-orientation of G satisfying the conclusion of Theorem 13.

Since G is the bipartite graph, let B and W be the partite sets of G. Since the torus is orientable, we can give a clockwise orientation at each point on the torus simultaneously. Suppose that $\{e_1, \ldots, e_m\}$ are the edges of $\vec{G}$ incident to a vertex v, where $\deg_G(v) = m$ and $\{e_1, \ldots, e_m\}$ appear around v in this clockwise order. Without loss of generality, we may assume that e_1 and e_k are two outgoing edges on v. Now we can put a segment l_v through a vertex v which locally separates all edges incident to v into two sets as shown in Fig. 3:

- if $v \in B$, then $e_1 \ldots e_{k-1}$ are located in one of the two sides separated by l_v, and all others are in the other side of l_v;
- if $v \in W$, then $e_2 \ldots e_k$ are located in one of the two sides separated by l_v, and all others are in the other side of l_v.

Next we consider whether we can glue l_v's for all $v \in V(\vec{G})$ to get a vertex-face m-family for some $m \geqslant 1$. Let us consider a quadrilateral face f of $\vec{G}$ bounded by a 4-cycle $bwb'w'$, where $b, b' \in B$ and $w, w' \in W$. We may assume that b, w, b' and w' appear in the clockwise order. Observe that depending on the direction of the edge $w'b$, exactly one of $l_{w'}$ and l_b intersects f. (If the edge $w'b$ is directed from w' to b, then whatever the directions of $w'b'$ and bw are, $l_{w'}$ intersects with f and l_b does not; otherwise l_b intersects with f and $l_{w'}$ does not. See Fig. 4.) Similarly, it follows that exactly one of $l_{b'}$ and l_w intersects f. Then we can find that exactly two segments intersect at each face, and glue them at a center of the face. Hence $\bigcup_{v \in V(\vec{G})} l_v$ form a set of several simple closed curves visiting each vertex and each face of $\vec{G}$ exactly once, but crossing no edge of $\vec{G}$ transversely. Therefore, $\bigcup_{v \in V(\vec{G})} l_v$ can be regarded as a vertex-face m-family $\mathcal{L}_m = \{l_1, \ldots, l_m\}$ for some $m \geqslant 1$.

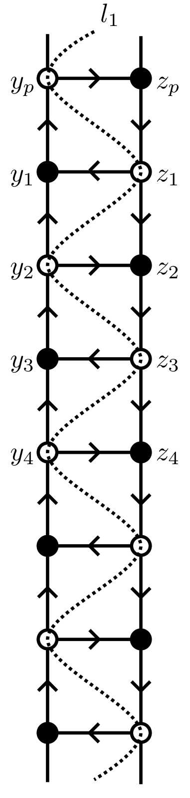

Fig. 5. The curve l_1 for Q'_C.

In [9], it is proved that each l_i is essential in the torus. Then we can take the indices of $l_1, \dots, l_m$ so that l_i and l_{i+1} bound an annulus A_i not containing any other l_j. Moreover, when $m \geqslant 2$, it has been proved in [9] that for any i, changing the orientation of some edges in A_i, we can take a simple closed curve l'_i such that $\{l_1, \dots, l_{i-1}, l'_i, l_{i+2}, \dots, l_m\}$ is a vertex-face $(m-1)$-family in the new 2-orientation of G. Therefore, if $G = Q$, then by iteration of changing the orientation, we can find a vertex-face curve for Q.

Now consider the case where $G = Q'_C$. Let A be the annulus bounded by Y and Z and containing the edges $y_i z_i$'s. Here we assume that y_1 is black, since the proof holds regardless of color choice. Then, by the rules for the arrangement of segments and the way to glue them, A contains a single simple closed curve, say l_1, in a vertex-face m-family $\{l_1, \dots, l_m\}$ for some $m \geqslant 1$, passing through all z_i with i odd, all y_i with i even and no others. See Fig. 5. Since Q'_C has faces not contained in A, we have $m \geqslant 2$. If $m \geqslant 3$, then we can merge l_2 and l_3 to get a vertex-face $(m-1)$-family containing l_1. This process can be repeated until $m = 2$. If $m = 2$, then contracting all edges $y_i z_i$ for $i = 1, \dots, m$ to eliminate l_1, we get a vertex-face curve for Q as shown in Fig. 6, since l_2 passes through all vertices and all faces of Q exactly once. $\square$

5. Proof of the theorems

We first prove Theorem 13.

Proof of Theorem 13. If C is a cycle and P is a path which meets C exactly in its end-vertices, then we say that P is a C-path. For an essential cycle C, a C-path P is called *an essential C-path* if no cycle

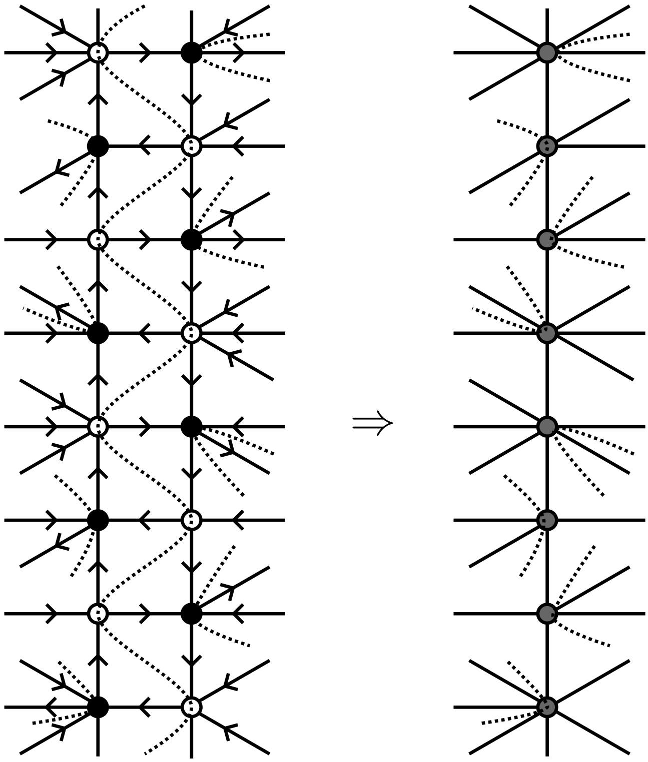

Fig. 6. A vertex-face curve in Q.

in $C \cup P$ bounds a 2-cell on F^2. Especially in case of $|E(P)| = 1$, we say that P is *an essential C-edge*. For two vertices x and y in a graph G, let $d_G(x, y)$ denote the distance of x and y in G.

Let $\mathcal{C}$ be the set of shortest essential even cycles of Q. By Lemma 11, $\mathcal{C} \neq \emptyset$. Let $C = v_1 v_2 \dots v_p$ be a cycle in $\mathcal{C}$. In this section, we always consider the subscript of v as modulo p. Since Q is a quadrangular map, it follows that if $v_i v_j \in E(Q) \setminus E(C)$, then $v_i v_j$ is an essential C-edge. Moreover, the cycle $v_i v_{i+1} \dots v_j v_i$ is an odd cycle, since otherwise we would get an essential even cycle shorter than C, which is a contradiction. Then we have the following:

$$\begin{aligned}&\text{if } v_i v_j \in E(Q) \setminus E(C)\text{, then } v_i v_j \text{ is an essential } C\text{-edge}\\ &\text{and the parities of } i \text{ and } j \text{ are the same.}\end{aligned} \tag{1}$$

Here we fix an orientation of C and let $E_R(C, v_i)$ and $E_L(C, v_i)$ denote the set of edges in $E(Q) \setminus E(C)$ which are adjacent to v_i on the right-hand side and left-hand side of C, respectively. We define six sets of pairs of indices as follows:

$$\begin{aligned}
\Lambda_B(C) = {}&\bigl\{(i, j) \bigm| i, j\text{: odd},\ v_i v \in E_L(C, v_i),\ v_j v \in E_L(C, v_j) \text{ for some } v \in V(C)\bigr\}\\
&\cup \bigl\{(i, j) \bigm| i, j\text{: even},\ v_i v \in E_R(C, v_i),\ v_j v \in E_R(C, v_j) \text{ for some } v \in V(C)\bigr\};\\
\Lambda_W(C) = {}&\bigl\{(i, j) \bigm| i, j\text{: odd},\ v_i v \in E_R(C, v_i),\ v_j v \in E_R(C, v_j) \text{ for some } v \in V(C)\bigr\}\\
&\cup \bigl\{(i, j) \bigm| i, j\text{: even},\ v_i v \in E_L(C, v_i),\ v_j v \in E_L(C, v_j) \text{ for some } v \in V(C)\bigr\};\\
A_B(C) = {}&\bigl\{(i, j) \bigm| i, j\text{: odd},\ v_i v \in E_L(C, v_i),\ v_j v \in E_L(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\bigr\}\\
&\cup \bigl\{(i, j) \bigm| i, j\text{: even},\ v_i v \in E_R(C, v_i),\ v_j v \in E_R(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\bigr\};
\end{aligned}$$

$$A_W(C) = \{(i,j) \mid i,j\text{: odd},\ v_iv \in E_R(C,v_i),\ v_jv \in E_R(C,v_j) \text{ for some } v \in V(Q) \setminus V(C)\}$$
$$\cup \{(i,j) \mid i,j\text{: even},\ v_iv \in E_L(C,v_i),\ v_jv \in E_L(C,v_j) \text{ for some } v \in V(Q) \setminus V(C)\};$$
$$X_B(C) = \{(i,j) \mid i\text{: odd},\ j\text{: even},\ v_iv \in E_L(C,v_i),\ v_{i+2}v \in E_L(C,v_{i+2}),$$
$$v_jv \in E_R(C,v_j) \text{ and } v_{j+2}v \in E_R(C,v_{j+2}) \text{ for some } v \in V(Q) \setminus V(C)\};$$
$$X_W(C) = \{(i,j) \mid i\text{: even},\ j\text{: odd},\ v_iv \in E_L(C,v_i),\ v_{i+2}v \in E_L(C,v_{i+2}),$$
$$v_jv \in E_R(C,v_j) \text{ and } v_{j+2}v \in E_R(C,v_{j+2}) \text{ for some } v \in V(Q) \setminus V(C)\}.$$

By the minimality of C, $d_C(v_i,v_j) = 2$ holds for every $(i,j) \in A_B(C) \cup A_W(C)$, that is, $j = i+2$ or $i = j+2$. For $(i,i+2) \in A_W(C)$, we take $v'_{i+1} \in N(v_i) \cap N(v_{i+2})$ such that $v_iv_{i+1}v_{i+2}v'_{i+1}v_i$ bounds a 2-cell so that $v_iv'_{i+1}$ is the rightmost edge when i is odd, and $v_iv'_{i+1}$ is the leftmost edge when i is even. Let

$$C' = v_1v_2 \ldots v_iv'_{i+1}v_{i+2}v_{i+3} \ldots v_1. \tag{2}$$

When i is odd (resp. even), there is no common neighbor of v_i and v_{i+2} in the right-hand (resp. left-hand) side of C' by the choice of v'_{i+1}. Moreover, there is no common neighbor of v_{i-1} and v'_{i+1} in the left-hand (resp. right-hand) side of C' and no common neighbor of v'_{i+1} and v_{i+3} in the left-hand (resp. right-hand) side of C', since every edge in $E_L(C',v'_{i+1})$ (resp. $E_R(C',v'_{i+1})$) is contained in the 2-cell bounded by $v_iv'_{i+1}v_{i+2}v_{i+1}$. Therefore, the following hold:

$$A_W(C') \subseteq A_W(C) \setminus \{(i,i+2)\}, \tag{3}$$
$$X_W(C') \subseteq X_W(C) \quad \text{and} \tag{4}$$
$$X_W(C) = \emptyset \quad \Rightarrow \quad \Lambda_W(C') \subseteq \Lambda_W(C). \tag{5}$$

Here we may assume that $C \in \mathcal{C}$ was chosen so that:

(i) $|A_W(C)|$ is as small as possible and
(ii) $|\Lambda_W(C)|$ is as small as possible, subject to (i).

Now we prove $\Lambda_W(C) = A_W(C) = \emptyset$. If $(i,i+2) \in A_W(C)$, we can take the new cycle C' as in (2). Then it follows from (3) that $|A_W(C')| < |A_W(C)|$, which is a contradiction. Hence $A_W(C) = \emptyset$ holds.

Next we shall prove $\Lambda_W(C) = \emptyset$. We need the following claim.

Claim 1. *Let $v_sv_{s'}$ be an essential C-edge and P be an essential C-path joining v_t and v'_t, where $s \neq t$, $s' \neq t'$ and $t \neq t'$. Then:*

(I) *If v_s, v_t, $v_{s'}$, $v_{t'}$ appear in C in this order or in the order v_s, $v_{t'}$, $v_{s'}$, v_t (possibly, $s = t'$ and/or $s' = t$), then $d_C(v_s,v_t) + d_C(v_{s'},v_{t'}) \leqslant |E(P)| + 1$.*
(II) *If v_s, v_t, $v_{s'}$, $v_{t'}$ appear in C in an order other than* (I), *then $d_C(v_s,v_{s'}) \leqslant |E(P)|$.*

Proof. Assume, to the contrary, that $d_C(v_t,v_s) + d_C(v_{t'},v_{s'}) \geqslant |E(P)| + 2$ in the case (I) or $d_C(v_s,v_{s'}) \geqslant |E(P)| + 1$ in the case (II). Without loss of generality, we may assume that $s \leqslant s', t, t'$. Here we define an essential cycle C' as follows:

(I-1) if $t \leqslant s' < t'$, then let $C' = v_1v_2 \ldots v_sv_{s'}v_{s'-1} \ldots v_tPv_{t'}v_{t'+1} \ldots v_1$;
(I-2) if $t' < s' \leqslant t$, then let $C' = v_sv_{s+1} \ldots v_{t'}Pv_tv_{t-1} \ldots v_{s'}v_s$;
(II-1) if $s' < t < t'$, then let $C' = v_1v_2 \ldots v_sv_{s'}v_{s'+1} \ldots v_tPv_{t'}v_{t'+1} \ldots v_1$;
(II-2) if $s' < t' < t$, then let $C' = v_1v_2 \ldots v_sv_{s'}v_{s'+1} \ldots v_{t'}Pv_tv_{t+1} \ldots v_1$;
(II-3) if $t < t' < s'$, then let $C' = v_sv_{s+1} \ldots v_tPv_{t'}v_{t'+1} \ldots v_{s'}v_s$;
(II-4) if $t' < t < s'$, then let $C' = v_sv_{s+1} \ldots v_{t'}Pv_tv_{t+1} \ldots v_{s'}v_s$.

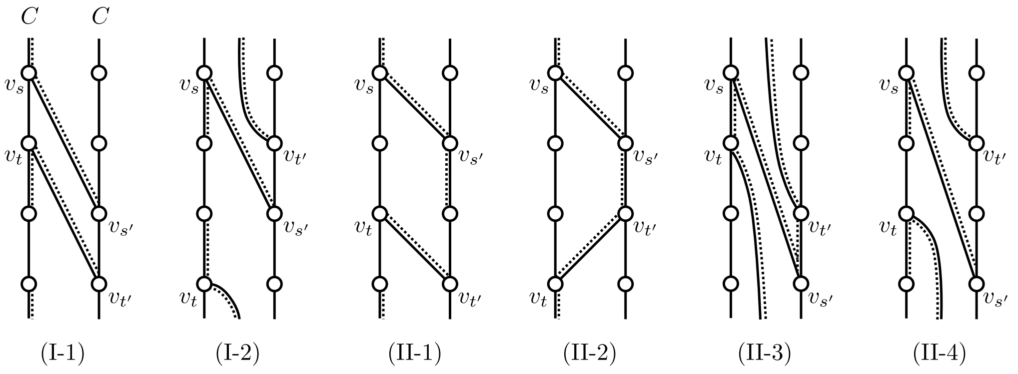

Fig. 7. The cycles C' shorter than C.

(See Fig. 7.) Note that C' in the case (I-1) is an even cycle since C' is obtained by the symmetric difference of C and the trail $v_s v_{s+1} \dots v_t P v_{t'} v_{t'-1} \dots v_{s'} v_s$ bounding a 2-cell, which has even length. Similarly, note that C' in all cases is an even cycle. Since for $i < j$, $d_C(v_i, v_j) \leqslant j - i$ even when $d_C(v_i, v_j) = p + i - j$, it follows from the minimality of C that

$$|C'| \leqslant \begin{cases} |C| - d_C(v_s, v_t) - d_C(v_{s'}, v_{t'}) + |E(P)| + 1 & \text{in the cases (I-1) and (I-2),} \\ |C| - d_C(v_s, v_{s'}) - d_C(v_t, v_{t'}) + |E(P)| + 1 & \text{in the cases (II-1)–(II-4).} \end{cases}$$

Since $d_C(v_t, v_s) + d_C(v_{t'}, v_{s'}) \geqslant |E(P)| + 2$ in the cases (I-1) and (I-2), and since $d_C(v_s, v_{s'}) \geqslant |E(P)| + 1$ and $d_C(v_t, v_{t'}) \geqslant 1$ in the cases (II-1)–(II-4), C' is shorter than C. Thus we obtain a contradiction in each case. □

Assume $\Lambda_W(C) \neq \emptyset$. Then, without loss of generality, we may assume that there exist three vertices v_k, v_l and $v_{k'}$ in $V(C)$ such that $v_k v_{k'} \in E_R(C, v_k)$ and $v_l v_{k'} \in E_R(C, v_l)$ for some odd integers k and l with $k < l$. Also we may assume that the cycle $v_k v_{k+1} \dots v_l v_{k'} v_k$ bounds a 2-cell on the torus. Since Q has no loop, v_k, v_l and $v_{k'}$ are distinct. By (1), k' is also odd. If $k < k' < l$, then $v_1 v_2 \dots v_k v_{k'} v_l v_{l+1} \dots v_1$ is an essential even cycle which is shorter than C, a contradiction. Hence $k' < k$ or $l < k'$ holds. Without loss of generality, we may assume that $l < k'$. Then we have $k' \geqslant k + 4$ and $k' \geqslant l + 2$.

Using Claim 1, we consider possible positions of essential C-edges and essential C-paths, in order to prove $\Lambda_B(C) = \emptyset$ and $X_B(C) = \emptyset$ in the following two paragraphs respectively.

Suppose that $\Lambda_B(C) \neq \emptyset$. Then there exists an essential C-edge $v_m v_{m'}$ with $v_m v_{m'} \in E_R(C, v_m)$ and $m' \neq k'$. Recall that the parities of m and m' are the same. Now consider Claim 1 for $s = k$ (or $s = l$), $s' = k'$, $t = m$ and $t' = m'$. Since $d_C(v_k, v_{k'}) \geqslant 2$ and $d_C(v_l, v_{k'}) \geqslant 2$, the case (II) does not occur in either case, and hence $k = m$ or $d_C(v_k, v_m) + d_C(v_{k'}, v_{m'}) \leqslant 2$, and $l = m$ or $d_C(v_l, v_m) + d_C(v_{k'}, v_{m'}) \leqslant 2$. Since $k' \neq m'$, in either case, we have $d_C(v_k, v_m) \leqslant 1$ and $d_C(v_l, v_m) \leqslant 1$. If m is odd, then $d_C(v_k, v_m) = d_C(v_l, v_m) = 0$, contradicting $v_k \neq v_l$. If m is even, then $d_C(v_k, v_m) = d_C(v_l, v_m) = 1$. Since the cycle $v_k v_{k+1} \dots v_l v_{k'} v_k$ bounds a 2-cell on the torus, we have $m \leqslant k$ or $m \geqslant l$, and hence $m = k - 1 = l + 1$. However, this implies that $k = 1$ and $l = p - 1$, and hence $k < k' < l$, a contradiction. These imply that there exists no essential C-edge $v_m v_{m'}$ with $v_m v_{m'} \in E_R(C, v_m)$ and $m' \neq k'$, and hence $\Lambda_B(C) = \emptyset$.

Next suppose that $X_B(C) \neq \emptyset$. Then there exists an essential C-path $v_m v v_{m'}$ of length 2 such that $v_m v \in E_R(C, v_m)$, m is even, and $m' \neq k'$. Note that the parities of m and m' are different, so $d_C(v_{k'}, v_{m'}) \geqslant 2$. Notice also that $|C| \geqslant 6$ because k, l and k' are pairwise distinct odd integers. Since the cycle $v_k v_{k+1} \dots v_l v_{k'} v_k$ bounds a 2-cell on the torus, we have $m < k$ or $m > l$. Now consider Claim 1 for $s = k$ ($s = l$), $s' = k'$, $t = m$ and $t' = m'$. If the case (I) occurs in case when $s = k$, then $d_C(v_k, v_m) + d_C(v_{k'}, v_{m'}) \leqslant 3$. Since k is odd and m is even, we have that $m = k - 1$ and $m' = k' - 2$ or $k' + 2$. Similarly, if the case (I) occurs in case when $s = l$, then $m = l + 1$ and $m' = k' - 2$ or $k' + 2$.

Assume that the case (II) occurs for both cases $s = k$ and $s = l$. Then $d_C(v_k, v_{k'}) \leqslant 2$ and $d_C(v_l, v_{k'}) \leqslant 2$, that is, $k' = k - 2 = l + 2$. Since $m < k$ or $m > l$, we obtain $m = k' \pm 1$. If $m' \neq k' - 2, k' + 2$, then v_k, $v_{m'}$, $v_{k'}$, v_m (when $m = k' + 1$) or v_l, v_m, $v_{k'}$, $v_{m'}$ (when $m = k' - 1$) appear in C in this order, and the case (I) holds, a contradiction. Thus, we have $m' = k' - 2$ or $k' + 2$. In any of the above situations, $m = k - 1$ or $l + 1$, and $m' = k' - 2$ or $k' + 2$. This implies that if $(n', n) \in X_B(C)$, then we have $n = l + 1$, $n + 2 = k - 1$. Moreover, either $n' = k' - 2$, $n' = k'$ or $n' = k' + 2$ and $n' + 2 = k' - 2$ (the last case occurs only when $p = 6$). Recall that $k' = k - 2 = l + 2$. Let v be the vertex such that $vv_n, vv_{n+2}, vv_{n'}, vv_{n'+2} \in E(Q)$. In the case $n' = k' - 2$ or $n' = k'$, $v_l v_n v v_{k'} v_l$ or $v_k v_{k'} v v_{n+2} v_k$ is an essential even cycle of length four, contradicting the choice of C. In the case $n' = k' + 2$ and $n' + 2 = k' - 2$, $v_k v_{k'} v_n v v_k$ is an essential even cycle of length four, contradicting the choice of C. Thus, $X_B(C) = \emptyset$.

By changing the parities of the indices of C, we can find the cycle C_1 with $\Lambda_W(C_1) = X_W(C_1) = \emptyset$. If $A_W(C_1) = \emptyset$, then C_1 is a desired cycle. If $A_W(C_1) \neq \emptyset$, then it follows from (3) that we can decrease the value $|A_W(C_1)|$ by taking a new cycle as in (2). Iterating this operation, we obtain an essential even cycle C_1' such that $A_W(C_1') = \emptyset$. Moreover, since $X_W(C_1) = \emptyset$, it follows from (4) and (5) that $\Lambda_W(C_1') \subseteq \Lambda_W(C_1) = \emptyset$. Consequently we have $\Lambda_W(C) = A_W(C) = \emptyset$.

Recall that Q_C is the annulus quadrangulation appeared in Section 4, which is obtained from Q by cutting along C. Let $Y = y_1 y_2 \ldots y_p$ and $Z = z_1 z_2 \ldots z_p$ be the cycles corresponding to two boundary components of Q_C, where y_i and z_i correspond to the same vertex v_i in Q, for each i. We may assume that $E_{Q_C}(y_i) - E(Y) = E_L(C, v_i)$ for any $y_i \in Y$, where $E_{Q_C}(y_i)$ is the set of edges incident with y_i in Q_C. Since Q is a quadrangulation and C is an essential even cycle, Q_C can be regarded as a map on the annulus with each face bounded by an even cycle. Hence Q_C is a bipartite graph. Let

$$Y_W = \bigcup_{i:\text{ even}} y_i, \qquad Y_B = \bigcup_{i:\text{ odd}} y_i, \qquad Z_W = \bigcup_{i:\text{ odd}} z_i, \qquad Z_B = \bigcup_{i:\text{ even}} z_i$$

and let

$$\varphi(v) = \begin{cases} 0 & \text{for } v \in Y_W \cup Z_W, \\ 1 & \text{for } v \in Y_B \cup Z_B, \\ 2 & \text{for } v \notin Y \cup Z. \end{cases}$$

Let $G = Q_C - (E(Y) \cup E(Z))$. Let W and B be a bipartition of $V(G)$, referred as *white* and *black* vertices, where we suppose $Y_W, Z_W \subset W$ and $Y_B, Z_B \subset B$. By (1), each of Y and Z is an independent set in G.

Claim 2. *There exists an orientation of G such that $od(v) = \varphi(v)$ for any $v \in V(G)$.*

Proof. By Euler's formula, we have $|E(Q)| = 2|V(Q)|$. Thus $|E(G)| = |E(Q_C)| - 2p = |E(Q)| - p = 2|V(Q)| - p = 2|V(Q_C)| - 3p = 2|V(G)| - 3p = \sum_{v \in V(G)} \varphi(v)$. By Proposition 10, it suffices to prove $|E([S])| \leqslant \sum_{v \in S} \varphi(v)$ for every $S \subset V(G)$ with $[S]$ connected.

Case 1. $[S]$ contains an essential cycle on the annulus.

Observe that $[S]$ has two boundary walks $L_1 = a_1 a_2 \ldots a_{l_1} a_1$ and $L_2 = b_1 b_2 \ldots b_{l_2} b_1$ such that $S \cap Y \subset L_1$ and $S \cap Z \subset L_2$. Let $Y_B' = Y_B \cap L_1$, $Y_W' = Y_W \cap L_1$, $Z_B' = Z_B \cap L_2$ and $Z_W' = Z_W \cap L_2$. Note that $l_1 \geqslant |\{a_i, a_{i+1} \mid a_i \in Y_B'\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} \mid a_i \in Y_W'\}|$. Since Y is independent, $a_{i+1} \notin Y_B'$ holds for every $a_i \in Y_B'$. Moreover, for every $a_i \in Y_W'$, $\{a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}\} \cap Y_B' = \emptyset$ holds since G is bipartite and Y is independent, and $\{a_{i+1}, a_{i+2}, a_{i+3}\} \cap Y_W' = \emptyset$ holds since $\Lambda_W(C) = A_W(C) = \emptyset$. Therefore we have $l_1 \geqslant 2|Y_B'| + 4|Y_W'|$. Similarly, $l_2 \geqslant 2|Z_B'| + 4|Z_W'|$ holds. Thus by Euler's formula,

$$\begin{aligned} \left|E\left([S]\right)\right| &\leqslant 2|S| - \frac{l_1 + l_2}{2} \\ &\leqslant 2|S| - \left(\left|Y_B'\right| + \left|Z_B'\right| + 2\left(\left|Y_W'\right| + \left|Z_W'\right|\right)\right) \\ &= \sum_{v \in S} \varphi(v). \end{aligned}$$

Case 2. $[S]$ contains no essential cycle on the annulus.

Let $L = a_1 a_2 \dots a_l a_1$ be the boundary closed walk of $[S]$. Note that $S \cap (Y \cup Z) \subset L$. Let $Y'_B = Y_B \cap L$, $Y'_W = Y_W \cap L$, $Z'_B = Z_B \cap L$ and $Z'_W = Z_W \cap L$. Here we choose two subwalks $L_Y = a_1 \dots a_r$, $L_Z = a_s \dots a_t$ of L so that L_Y and L_Z are as short as possible subject to $1 \leqslant r < s \leqslant t$, $L \cap Y \subset L_Y$ and $L \cap Z \subset L_Z$.

Since Y is independent, $a_{i+1} \notin Y'_B$ holds for every $a_i \in Y'_B$. Moreover, for every $a_i \in Y'_W$, $\{a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}\} \cap Y'_B = \emptyset$ holds since G is bipartite and Y is independent, and $\{a_{i+1}, a_{i+2}, a_{i+3}\} \cap Y'_W = \emptyset$ holds since $\Lambda_W(C) = A_W(C) = \emptyset$. Now the vertices in $\{a_i, a_{i+1} \mid a_i \in Y'_B\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} \mid a_i \in Y'_W\}$ are contained in

- the segment $a_l a_1 a_2 \dots a_r a_{r+1} a_{r+2}$ of L when $a_1, a_r \in Y'_W$;
- the segment $a_l a_1 a_2 \dots a_r a_{r+1}$ of L when $a_1 \in Y'_W$ and $a_r \notin Y'_W$;
- the segment $a_1 a_2 \dots a_r a_{r+1} a_{r+2}$ of L when $a_r \in Y'_W$ and $a_1 \notin Y'_W$;
- the segment $a_1 a_2 \dots a_r a_{r+1}$ of L when $a_1, a_r \notin Y'_W$.

Hence we have $r + 1 + |\{a_1, a_r\} \cap Y'_W| \geqslant |\{a_i, a_{i+1} \mid a_i \in Y'_B\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} \mid a_i \in Y'_W\}| = 2|Y'_B| + 4|Y'_W|$, and thus $r \geqslant 2|Y'_B| + 4|Y'_W| - 1 - |\{a_1, a_r\} \cap Y'_W|$ holds. Similarly we obtain $t - s + 1 \geqslant 2|Z'_B| + 4|Z'_W| - 1 - |\{a_s, a_t\} \cap Z'_W|$. Consequently,

$$\begin{aligned} l &= r + (t - s + 1) + (l - t) + (s - 1 - r) \\ &\geqslant 2|Y'_B| + 4|Y'_W| - 1 - |\{a_1, a_r\} \cap Y'_W| \\ &\quad + 2|Z'_B| + 4|Z'_W| - 1 - |\{a_s, a_t\} \cap Z'_W| + (l - t) + (s - 1 - r). \end{aligned}$$

Here $l - t \geqslant 1$ when $|\{a_1\} \cap Y'_W| + |\{a_t\} \cap Z'_W| = 2$, and $s - 1 - r \geqslant 1$ when $|\{a_r\} \cap Y'_W| + |\{a_s\} \cap Z'_W| = 2$. Hence $-|\{a_1, a_r\} \cap Y'_W| - |\{a_s, a_t\} \cap Z'_W| + (l - t) + (s - 1 - r) \geqslant -2$, which implies $l \geqslant 4(|Y'_W| + |Z'_W|) + 2(|Y'_B| + |Z'_B|) - 4$. Thus by Euler's formula,

$$\begin{aligned} |E([S])| &\leqslant 2|S| - l/2 - 2 \\ &\leqslant 2|S| - \bigl(2(|Y'_W| + |Z'_W|) + |Y'_B| + |Z'_B|\bigr) \\ &= \sum_{v \in S} \varphi(v). \end{aligned}$$

This completes the proof of Claim 2. □

Consider the orientation of Q'_C which is obtained from the orientation of G constructed in Claim 2 and oriented edges $y_{i+1} y_i$, $z_i z_{i+1}$ for any integer i, $z_i y_i$ for any odd integer i, and $y_i z_i$ for any even integer i for this direction. Then this is an orientation of Q'_C required in the theorem, and hence we are done. □

Finally, we are ready to prove Theorem 2.

Proof of Theorem 2. Let G be a 3-connected bipartite torus quadrangulation one of whose partite sets consists of degree four vertices. By Proposition 6, there exists a quadrangular map Q with no contractible 2-cycle such that the radial graph of Q is isomorphic to G. Now observe that Theorem 8 has been proved by Theorem 13 and Proposition 14. So Q has a vertex-face curve, and hence, by Proposition 7, G is Hamiltonian. □

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